Solution Key | February 2019
Financial Econometrics A

Question A:

Consider the ARCH(1) model,

$$x_t = \sigma_t z_t, \tag{A.1}$$

with $z_t \sim i.i.d.N(0,1)$ and

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2, \quad \omega > 0, \;\; 1 > \alpha \geq 0.$$

Question A.1: Suppose that x_t is stationary with $E(x_t^2) < \infty$, and define $\gamma := E(x_t^2)$. Find an expression for γ in terms of ω and α, and show that σ_t^2 can be re-written as

$$\sigma_t^2 = \gamma(1-\alpha) + \alpha x_{t-1}^2. \tag{A.2}$$

Solution: Since x_t is stationary with $E(x_t^2) < \infty$, $E[x_t^2] = E[\sigma_t^2] = \omega + \alpha E[x_{t-1}^2]$. By stationarity, $\gamma := E[x_t^2] = E[x_{t-1}^2]$, such that $\gamma = \frac{\omega}{1-\alpha}$, where it is used that $0 \leq \alpha < 1$. Thus we obtain the reparametrization $\sigma_t^2 = \omega + \alpha x_{t-1}^2 = \gamma(1-\alpha) + \alpha x_{t-1}^2$.

Question A.2: Show that x_t is stationary and weakly mixing with $E(x_t^4) < \infty$ if $\alpha < 1/\sqrt{3}$. (Hint: Recall that $E[z_t^4] = 3$.)

Solution: x_t is a Markov chain with continuous transition density. Show that x_t satisfies drift criterion with drift function $\delta(x) = 1 + x^4$. Details should be given.

Question A.3: In the following, we define the vector of parameters in the model as $\theta = (\gamma, \alpha)'$. The log-likelihood contribution at time t, $l_t(\theta)$, in terms of (A.1) and (A.2) is (up to a scaling constant),

$$l_t(\theta) = -\log(\sigma_t^2(\theta)) - \frac{x_t^2}{\sigma_t^2(\theta)}, \quad \sigma_t^2(\theta) = \gamma(1-\alpha) + \alpha x_{t-1}^2.$$

Show that the score in the direction of α evaluated at the true parameters $\theta_0 = (\gamma_0, \alpha_0)$ is given by

$$\left.\frac{\partial l_t(\theta)}{\partial \alpha}\right|_{\theta=\theta_0} = \frac{\left(x_{t-1}^2 - \gamma_0\right)}{\gamma_0 + \alpha_0\left(x_{t-1}^2 - \gamma_0\right)}\left(z_t^2 - 1\right).$$

Solution: We have

$$\frac{\partial l_t(\theta)}{\partial \alpha} = \frac{\left(x_{t-1}^2 - \gamma\right)}{\sigma_t^2(\theta)} \left(\frac{x_t^2}{\sigma_t^2(\theta)} - 1\right),$$

such that

$$\left.\frac{\partial l_t(\theta)}{\partial \alpha}\right|_{\theta=\theta_0} = \frac{\left(x_{t-1}^2 - \gamma_0\right)}{\sigma_t^2(\theta_0)} \left(z_t^2 - 1\right) = \frac{\left(x_{t-1}^2 - \gamma_0\right)}{\gamma_0 + \alpha_0\left(x_{t-1}^2 - \gamma_0\right)} \left(z_t^2 - 1\right).$$

Question A.4: Assume that $0 < \alpha_0 < 1$. Show that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left.\frac{\partial l_t(\theta)}{\partial \alpha}\right|_{\theta=\theta_0} \xrightarrow{D} N(0, \xi), \quad \text{as } T \to \infty. \tag{A.3}$$

Explain briefly how (A.3) can be used.

Solution: Show (A.3) using CLT for weakly mixing processes. With $f_t = \frac{\left(x_{t-1}^2 - \gamma_0\right)}{\gamma_0 + \alpha_0\left(x_{t-1}^2 - \gamma_0\right)} \left(z_t^2 - 1\right)$, we have that

$$E[f_t | x_{t-1}] = 0.$$

It remains to show that

$$E[f_t^2] < \infty.$$

This is done by noting that

$$E[f_t^2] = E\left[\left(\frac{\left(x_{t-1}^2 - \gamma_0\right)}{\gamma_0 + \alpha_0\left(x_{t-1}^2 - \gamma_0\right)}\right)^2\right] E\left[\left(z_t^2 - 1\right)^2\right],$$

where $E\left[\left(z_t^2 - 1\right)^2\right] = 2$ and

$$\begin{aligned} & E\left[\left(\frac{\left(x_{t-1}^2 - \gamma_0\right)}{\gamma_0 + \alpha_0\left(x_{t-1}^2 - \gamma_0\right)}\right)^2\right] \\ & = E\left[\left(\frac{x_{t-1}^2}{(1-\alpha_0)\gamma_0 + \alpha_0 x_{t-1}^2}\right)^2 + \left(\frac{\gamma_0}{(1-\alpha_0)\gamma_0 + \alpha_0 x_{t-1}^2}\right)^2 - \frac{2x_{t-1}^2\gamma_0}{((1-\alpha_0)\gamma_0 + \alpha_0 x_{t-1}^2)^2}\right], \end{aligned}$$

with

$$E\left[\left(\frac{x_{t-1}^2}{(1-\alpha_0)\gamma_0+\alpha_0 x_{t-1}^2}\right)^2\right] \leq \alpha_0^{-2} < \infty,$$
$$E\left[\left(\frac{\gamma_0}{(1-\alpha_0)\gamma_0+\alpha_0 x_{t-1}^2}\right)^2\right] \leq (1-\alpha_0)^{-2} < \infty,$$
$$E\left[\left|\frac{-2x_{t-1}^2\gamma_0}{((1-\alpha_0)\gamma_0+\alpha_0 x_{t-1}^2)^2}\right|\right] \leq (\alpha_0(1-\alpha_0))^{-1} < \infty.$$

(A.3) can be used for deriving the (limiting) distribution of the MLE. This can be used for addressing the estimation uncertainty of the model parameters, or hypothesis testing. Note that additional condtions are needed. Some details should be provided.

Question A.5: Suppose that one seeks to investigate whether the level, γ, of the volatility is beyond some given value, γ_0, i.e. $\gamma > \gamma_0$. This can be done by testing the null hypothesis

$$H_0 : \gamma = \gamma_0,$$

against the alternative $H_A : \gamma > \gamma_0$. Explain how you would test H_0 based on the maximum likelihood estimator for $\theta = (\gamma, \alpha)'$. Be specific about which conditions are needed.

Solution: Suppose that the MLE, $\hat{\theta} = (\hat{\gamma}, \hat{\alpha})$, satisfies $\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{D} N(0, \Omega)$ for some covariance matrix Ω. Ideally, it is mentioned when this is the case (here one could refer to Question A.4). With $\hat{\Omega}_{\gamma\gamma}$ a consistent estimator for the element at first row and first column of Ω, one may construct the t-statistic $t_{\gamma=\gamma_0} = \frac{\hat{\gamma}-\gamma_0}{\hat{\Omega}_{\gamma\gamma}/\sqrt{T}}$, which should be approximately standard normal. Then the hypothesis could be tested using a standard one-sided t-test based on $t_{\gamma=\gamma_0}$.

Question B:

Consider the log-returns of a portfolio y_t given in Figure B.1 with $t = 1, 2, ..., T = 1000$.

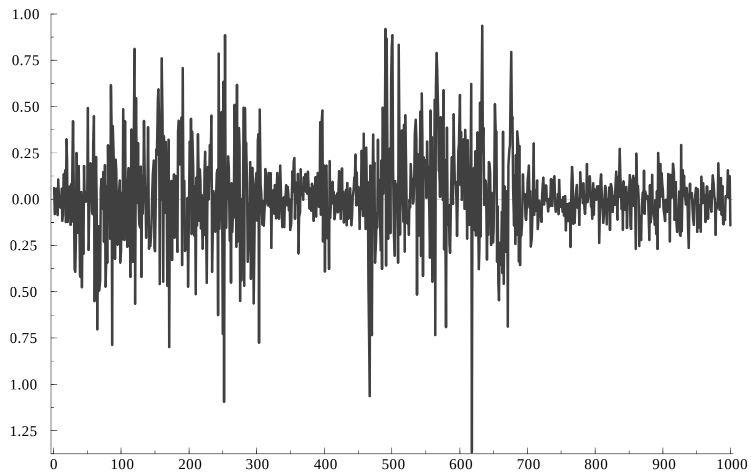

Figure B.1: Portfolio returns, y_t

Question B.1: Estimation with a 2-state Markov switching stochastic volatility model, gave the following output in the usual notation in terms of the transition matrix $P = (p_{ij})_{i,j=1,2}$ and smoothed standardized residuals, $\hat{z}_t^* = y_t / \hat{E}[\sigma_t | y_1, ..., y_T]$:

$\hat{P}$, QMLE of P:	$\hat{p}_{11} = 0.97$ $\hat{p}_{22} = 0.99$
	p-values for tests based on $\hat{z}_t^*$:
Normality test:	0.06
LM-test for no ARCH:	0.10

What would you conclude on the basis of the output and the graph?

Solution: Misspecification tests based on $\hat{z}_t^*$ indicate that the model is well-specified. Point estimates of transition probabiliaties p_{11} and p_{22} indicate that the regimes, and hence the volatility σ_t is highly persistent.

Question B.2: In order to compute the Value-at-Risk (VaR) of the portfolio, the following ARCH-type model was proposed:

$$y_t = \sigma_{s_t,t} z_t \tag{B.1}$$

Here the process (z_t) is $i.i.d.N(0,1)$ and independent of the $i.i.d.$ process (s_t), $s_t \in \{1,2\}$, where $p = P(s_t = 1) = 1 - P(s_t = 2)$. Moreover,

$$\sigma_{1,t}^2 = \omega + \alpha y_{t-1}^2 \quad \text{and} \quad \sigma_{2,t}^2 = \gamma. \tag{B.2}$$

Thus the parameters of the model are $\theta = (\omega, \alpha, \gamma, p)$ with $\omega, \gamma > 0$, $\alpha \geq 0$ and $p \in [0,1]$.

Provide a brief interpretation of the model.

Suppose, first that $(s_1, ..., s_T)$ is observed. Then, based on $(y_0, y_1, ..., y_T, s_1, ..., s_T)$, the log-likelihood function, $L_T(\theta)$, conditional on the initial value y_0 is given by

$$L_T(\theta) = \log f_\theta(y_1, ..., y_T, s_1, ..., s_T | y_0).$$

Argue that

$$f_\theta(y_1, ..., y_T, s_1, ..., s_T | y_0) = \prod_{t=1}^{T} f_\theta(y_t | s_t, y_{t-1}) p_\theta(s_t),$$

and provide expressions for $f_\theta(y_t | s_t, y_{t-1})$ and $p_\theta(s_t)$.

Solution: Two-state model. x_t is ARCH(1) in state 1, and Gaussian noise in state 2.

The expression $f_\theta(y_1, ..., y_T, s_1, ..., s_T | y_0) = \prod_{t=1}^{T} f_\theta(y_t | s_t, y_{t-1}) p_\theta(s_t)$ is obtained through straightforward factorization, using that $f_\theta(y_t | y_0, .., y_{t-1}, s_1, ..., s_t) = f_\theta(y_t | s_t, y_{t-1})$ and $p_\theta(s_t | y_0, ..., y_{t-1}, s_1, ..., s_{t-1}) = p_\theta(s_t)$. We have that

$$\begin{aligned} f_\theta(y_t | s_t, y_{t-1}) &= f_\theta(y_t | s_t = 1, y_{t-1})^{1(s_t=1)} f_\theta(y_t | s_t = 2, y_{t-1})^{1(s_t=2)}, \\ p_\theta(s_t) &= P(s_t = 1)^{1(s_t=1)} P(s_t = 2)^{1(s_t=2)}, \end{aligned}$$

with

$$\begin{aligned} f_\theta(y_t | s_t = 1, y_{t-1}) &= \frac{1}{\sqrt{2\pi(\omega + \alpha y_{t-1}^2)}} \exp\left(-\frac{y_t^2}{2(\omega + \alpha y_{t-1}^2)}\right), \\ f_\theta(y_t | s_t = 2, y_{t-1}) &= \frac{1}{\sqrt{2\pi\gamma}} \exp\left(-\frac{y_t^2}{2\gamma}\right), \\ P(s_t = 1) &= 1 - P(s_t = 2) = p. \end{aligned}$$

Question B.3: Next, suppose that s_t is unobserved. Then one may instead of $L_T(\theta)$ consider the function

$$L_T^\dagger(\theta) = \sum_{t=1}^{T} p_t^\dagger \left\{ \log f_\theta(y_t|y_{t-1}, s_t = 1) + \log(p) \right\} + \sum_{t=1}^{T} (1 - p_t^\dagger) \left\{ \log f_\theta(y_t|y_{t-1}, s_t = 2) + \log(1 - p) \right\},$$

where

$$p_t^\dagger = P_{\theta^\dagger}[s_t = 1|y_0, y_1, ..., y_T]$$

for some fixed $\theta^\dagger$.

Discuss briefly estimation of θ based on $L_T^\dagger(\theta)$.

Solution: Use EM-algorithm. Details should be provided.

Question B.4: The 5% level Value at Risk, $\text{VaR}_{T+1}^{5\%}$, satisfies:

$$P\left(y_{T+1} \leq -\text{VaR}_{T+1}^{5\%} \mid y_0, y_1, ..., y_T\right) = 5\%.$$

Suppose that p is known and fixed with $p = 1$. Then

$$\text{VaR}_{T+1}^{5\%} = -\sigma_{1,T+1}\Phi^{-1}(0.05),$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard normal distribution.

Explain briefly how you would compute an estimate of $\text{VaR}_{T+1}^{5\%}$.

Explain briefly how you would compute an estimate of $\text{VaR}_{T+1}^{5\%}$ if instead $p = 1/2$.

Solution: If $p = 1$, y_{T+1} is ARCH(1). Given a sample $(y_0, .., y_T)$, obtain parameter estimates by MLE, $(\hat{\omega}, \hat{\alpha})$. Compute $\widehat{\text{VaR}_{T+1}^{5\%}} = -\hat{\sigma}_{1,T+1}\Phi^{-1}(0.05)$, with $\hat{\sigma}_{1,T+1} = \sqrt{\hat{\omega} + \hat{\alpha}y_T^2}$.

If $p = 1/2$, one may compute an estimate of $\text{VaR}_{T+1}^{5\%}$ by simulations. Specifically, given a sample $(y_0, .., y_T)$, obtain estimates of (ω, α, γ) using EM-algorithm (Question B.3). Denote these estimates $(\tilde{\omega}, \tilde{\alpha}, \tilde{\gamma})$, and obtain

$$\tilde{\sigma}_{1,T+1}^2 = \tilde{\omega} + \tilde{\alpha}y_T^2 \quad \text{and} \quad \tilde{\sigma}_{2,T+1}^2 = \tilde{\gamma}.$$

Let $s_{T+1,1}, ..., s_{T+1,n}$ denote n independent draws from the distribution of s_{T+1} (i.e. $P(s_{T+1} = 1) = P(s_{T+1} = 2) = 1/2$), and let $z_{T+1,1}, ..., z_{T+1,n}$ denote n independent draws from $N(0, 1)$. Obtain $y_{T+1,i} = \tilde{\sigma}_{s_{T+1,i},T+1}^2 z_{T+1,i}$ for $i = 1, ..., n$. An estimate of $\text{VaR}_{T+1}^{5\%}$ can be obtained as minus the 5% quantile of $(y_{T+1,i} : i = 1, ..., n)$.